Entanglement entropy in stabilizer formalism with applications

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I Introduction

Entanglement is a unique product of quantum mechanics, which gives rise to various "spooky" phenomenon that do not have classical counter part. Entanglement spectrum and entanglement entropy are windows that let people to peek through. They serve as an important physical source which have rich applications in quantum cryptography, quantum computation, and condensed matter physics.

A wide class of interesting quantum entangled states can be described by so-called "stablizer formalism". This language has been widely used in quantum error correction, fault-tolerate quantum computing, and modern condensed matter physics, for example, the 2D toric code, 3D fracton models, and etc. I also want to mention that in strongly disorder systems and many-body localization systems(MBL), the Hamiltonian can be effectively described by a set of local integral of motions(LIOMs). Those LIOMs can also be viewed as stabilizer to stablize the quantum state. From this perspective, MBL effective Hamiltonian is also a stabilizer Hamiltonian.

In this note, we are going to review how to calculate entanglement entropy in the stabilizer formalism.

II Stabilizer formalism

Without loss of generality, here, and in the following, we will focus on qubit systems. Quantum states are vectors that live in the Hilbert space. For quantum states that admit a stabilizer formalism, they can also be described by a set of Pauli operators. Let's begin with a simple system that composes two qubits. The dimension of the Hilbert space is $2^2 = 4$. Let's have a set of Pauli operators: $S = \{XX, ZZ\}$. First of all, we notice that elements in set S mutually commute. And they all have eigenvalues ± 1. Therefore, they divide the Hilbert space into four subspaces that can be characterized by eigenvalues $(+1, +1), (+1, -1), (-1, +1), (-1, -1)$, and dimension of each subspace is one. Hence, we can use set S to denote the following four vectors/quantum states(bell states):

- $|\psi_1\rangle = |(0,0)\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle : E_{XX} = 1, E_{ZZ} = 1$
- $|\psi_2\rangle = |(0,1)\rangle = |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle : E_{XX} = -1, E_{ZZ} = 1$
- $|\psi_3\rangle = |(1,0)\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle : E_{XX} = 1, E_{ZZ} = -1$
- $|\psi_4\rangle = |(1,1)\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle : E_{XX} = -1, E_{ZZ} = -1$

We see that set S contains two stabilizers(mutually commuting Pauli operators) that divide the Hilbert space into four subspace, and we can distinguish them with the values of $E_{XX,ZZ}$. Now let's construct the projection operator that project Hilbert space into those subspaces. First, set $S = \{XX, ZZ\}$ could generate an abelian pauli group $G = \{II, XX, ZZ, -YY\}$. We can use a vector $\vec{n} \in \{0,1\}^2$ to label the elements in group G by $g(\vec{n}) = g_1^{n_1} g_2^{n_2}$. One can easily check the projection operator that projects to those pure states can be expressed by

$$|\vec{k}\rangle\langle\vec{k}| = \frac{1}{|G|}\sum_{\vec{n}}(-1)^{\vec{k}\cdot\vec{n}} g(\vec{n}). \tag{II.1}$$

Now we are ready to describe stabilizer quantum states in a more formal language. Here we consider a model consist of a set of N qubits living on the vertices or edges of a simple graph, with q qubits per each vertex or edge. $N = q|V|$, where V is the set of graph. Let P be the Pauli group acting on N qubits. A *stabilizer set* $S \subset P$ is a subset of Pauli group comprised of mutually commuting operators, which satisfies $|S| \geq N$. This means there are as least as many stabilziers as qubits. Supp$(S) = V$, each element of V is acted upon non-trivially by as least one stabilizer in S. The quantum states $|\psi\rangle$ which are stabilized by $S = \{O_s\}$ satisfy:

$$O_s|\psi\rangle = |\psi\rangle\,,\ \forall O_s \in S. \tag{II.2}$$

They form the ground state manifold for the stabilizer code Hamiltonian,

$$H_s = -\sum_s J_s O_s,\ (J_s > 0). \tag{II.3}$$

Since all members of S mutually commute, stabilizers O_s in S multiplicatively generate an Abelian group $G = \{\Pi_{s\in F} O_s : F \in \mathbb{P}[S]\}$, where the power set $\mathbb{P}[S]$ of S is the set of all subsets of S. Since elements in S may not be independent, S may over-determine G. Let $d_G = \log_2|G|$, and $\{O_i\}_{i\leq d_G} \subseteq S$ be a complete independent generating set for G. Elements $g \in G$ can be labelled by a binary vector $\vec{n} = (n_1, n_2, \cdots, n_{dG}) \in \{0,1\}^{d_G}$

via

$$g(\vec{n}) = \Pi_{i=1}^{d_G} O_i^{n_i}. \tag{II.4}$$

With every vector $\vec{k} \in \{0,1\}^{d_G}$, we associate a projection operator,

$$P^{\vec{k}} = \frac{1}{|G|} \sum_{g(\vec{n}) \in G} (-1)^{\vec{k}\cdot\vec{n}} g(\vec{n}) \tag{II.5}$$

We can check that $(P^k)^2 = P^k$, and $g(\vec{n})P^k = (-1)^{\vec{k}\cdot\vec{n}} P^k$. This implies P^k is a projection operator that project to simultaneously eigenstates of all of G, and the eigenvalue subspace are labelled by $\vec{k}$. Clearly, $\vec{k} = 0$ labels ground state manifold. P^k is a pure state projection if and only if $d_G = N$. For some interesting case, such as toric code, if the system is on a topological non-trivial manifold, $d_G < N$. This implies the degeneracy for all state (ground state and excited state), the degeneracy is 2^{N-d_G}. We may refer this as topological degeneracy.

III Entanglement spectrum and entanglement entropy using stabilzier formalism

We see that stabilizer formalism gives us an easy way to describe certain interesting quantum states. And certainly for those states, stabilizer formalism also provides us a cheaper way to calculate entanglement spectrum/entropy.

From the previous section, we know that pure state density matrix can be written as

$$|\vec{k}\rangle \langle\vec{k}| = P^k = \frac{1}{|G|} \sum_{\vec{n}} (-1)^{\vec{k}\cdot\vec{n}} g(\vec{n}), \tag{III.1}$$

if $d_G < N$, we can add $N - d_G$ "logical" operators to achieve the above. Consider a bipartition (A,B) of the whole region.

$$\rho_A = Tr_B |\vec{k}\rangle \langle\vec{k}| = \frac{1}{|G|} \sum_{\vec{n}} (-1)^{\vec{k}\cdot\vec{n}} Tr_B g(\vec{n}) \tag{III.2}$$

Notice any element $g(\vec{n})$ that is not equal to identity on B must contain at least one X or Z acting in the region B. And the partial trace on B will result in zero. Therefore, non-zero contribution will only come from the group element operators $g(\vec{n})$ support only on region A. Notice $TrI_B = 2^{N_B}$, where N_B is the number of spins in region B.

Therefore, we can conclude

$$\rho_A = \frac{2^{N_B}}{|G|} \sum_{\vec{n}_A} (-1)^{\vec{k}_A \cdot \vec{n}_A} g(\vec{n_A}) = \frac{|G_A|}{2^{N_A}} P_A^{(\vec{k}_A)}. \tag{III.3}$$

Note that the group G_A only includes complete stabilizers in A, and stabilizers acrossing AB region will result in zero. Since ρ_A is a projector, its entanglement entropy is straightforward:

$$S_A = N_A - \log_2 |G_A|. \tag{III.4}$$

Since $|G| \sim 2^N$, sometimes one prefers not to list all the elements in group G and find G_A. Could we use the information of the generators of G? The answer is yes. The idea is mainly from literature[1]. Let P_A be the map that takes $g_A \otimes g_B \in G$ onto $g_A \otimes I_B$.
Theorem: The generators for stabilizer group G of a bipartite state can always be brought into the *canonical form*:

$$G = \langle a_i \otimes I_B, I_A \otimes b_j, g_k, \bar{g}_k \rangle. \tag{III.5}$$

The first two generators will generate G_A and G_B, which are local subgroup of G. The generators of G_{AB} are $p = p(G_{AB})$ anti-commuting pairs $(g_k, \bar{g}_k)$, where $P_A(g_k)$ commute will all generators of G, except $\bar{g}_k$, and $P_A(\bar{g}_k)$ commute with all the canonical generators of G except for g_k. Therefore,

$$S_A = p = \frac{|G_{AB}|}{2}. \tag{III.6}$$

So calculating entanglement entropy is equivalent to search of anti-commuting pairs in the projections on A or B of the generators of G. Equivalently, one can compute the rank of $P_A(G)$, which is the rank of a $n \times 2n$ matrix with elements in $\mathbb{Z}_2$[1]. This concludes the discussion of using stabilizer formalism to calculate entanglement entropy of a stabilzier state.

References

[1] D. Fattal, T. Cubitt, Y. Yamamoto, S. Bravyi, and I.L. Chuang. Entanglement in the stabilizer formalism. (2004)

[2] H. Ma, A.T. Schmitz, S.A. Parameswaran, M. Hermele, and R.M. Nandkishore. Topological entanglement entropy of fraction stabilizer code.(2018)

[3] A.T. Schmitz, S.J. Huang, A. Perm. Entanglement Spectra of Stabilizer Codes: A Window into Gapped Quantum Phases of Matter.(2019)