Simple argument on LSM theorem

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This note review the essential argument about LSM theorem made by Oshikawa:

In a quantum many body system with periodic boundary condition and well-define conserved particle number, a finite excitation gap is possible only when particle number per unit cell of ground state is an integer number. Otherwise, it could be **gapless, symmetry breaking**, or **topologically ordered**.

Argument:
Suppose we have a quantum many-body state lives on a lattice $L_x \times L_y$ with periodic boundary condition. We arrange the lattice in to a cylinder and magnetic flux ϕ in the $\hat{x}$ direction is piercing through the cylinder. In the simplest gauge choice, we could choose A_x=ϕ/L_x. Let's first review single particle in magnetic field and gauge transformation. If a particle with charge $q = -e$ in a gauge field $(\phi, \vec{A})$, then the Schodinger equation is:

$$\frac{1}{2m}\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2\psi = (E+e\phi)\psi \tag{.1}$$

If we do a gauge transformation: $\psi \rightarrow e^{-i\frac{e}{\hbar c}f}\psi$, then $\vec{A} \rightarrow \vec{A} - \nabla f$.

Suppose the system is in the ground state $|\psi_0\rangle$ when $\phi = 0$. We can choose $|\psi_0\rangle$ is also an eigenstate of the translation operator $T_x = \exp\left(-\frac{i}{\hbar}p_x\right)$, since $[T_x, H] = 0$. So $T_x|\psi_0\rangle = \exp(-\frac{i}{\hbar}P_0)|\psi_0\rangle$. We can adiabatically increase ϕ by a flux quanta $\phi_0 = hc/e$. And the ground state of $H(\phi_0)$ is $|\phi_0^{'}\rangle$. Usually $H(\phi)$ differs from $H(0)$ and has AB-effect as a physical different consequence. But $H(\phi_0)$ can be gauge transformed to $H(0)$.We can do a a gauge transformation $U = \exp\left(-\frac{2\pi i}{L}\sum_{\vec{r}}xn_{\vec{r}}\right)$, $\psi \rightarrow U\psi$, and $\vec{A} \rightarrow n\frac{hc}{eL} - \frac{hc}{eL}\sum_r n_{\vec{r}} = 0$. Therefore, $U^\dagger H(\phi_0)U = H(0)$, and $|\phi_0^{'}\rangle$ is mapped to $U|\phi_0^{'}\rangle$. Now we are going to prove that $U|\phi_0^{'}\rangle$ and $|\phi_0\rangle$ are different ground state. We are going to calculate $T_xU|\phi_0^{'}\rangle$.

$$T_xU|\phi_0^{'}\rangle = UU^\dagger T_xU|\phi_0^{'}\rangle = UT_xe^{-\frac{2\pi i}{L}N}|\phi_0^{'}\rangle(.2)$$

Therefore, the momentum of final state is $P_0 + \frac{2\pi N}{L}$. $N = \frac{p}{q}L_xL_y$, so $P_{\text{final}} = P_0 + 2\pi\frac{p}{q}L_y$. Because $\frac{p}{q}L_y$ is not a integer, $U|\psi_0^{'}\rangle$ and $|\psi_0\rangle$. One can repeat the argument q times to get q-degenerate ground state. The degeneracy may be arise from symmetry breaking or topological order. And it is too general to get more information on that.

References

[1] M. Oshikawa. PhysRevLett.84.1535(2000). Commensurability, Excitation Gap, and Topology in Quantum Many-Particle Systems on a Periodic Lattice.