Quantum error correction code

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Three qubit code & Nine qubit code

Three qubit code

In classical error correction, in order to protect some information, the easiest way is to copy the information multiple times. But in quantum realm, we have non-clone theorem, which states that we cannot perfectly copy a quantum state. Therefore, we need some other methods to protect the information.

Let’s first discuss the three-qubit code, which encode the logical qubit into three qubits and it is protected from single qubit flip error.

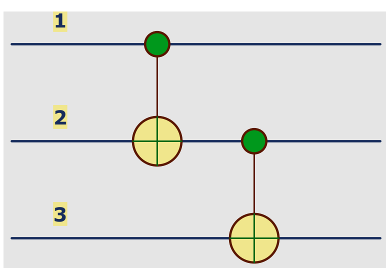
The two logical basis states $|0\rangle_L$ and $|1\rangle_L$ are defined as

$$\begin{aligned} |0\rangle_L &= |000\rangle, \\ |1\rangle_L &= |111\rangle. \end{aligned} \tag{1}$$

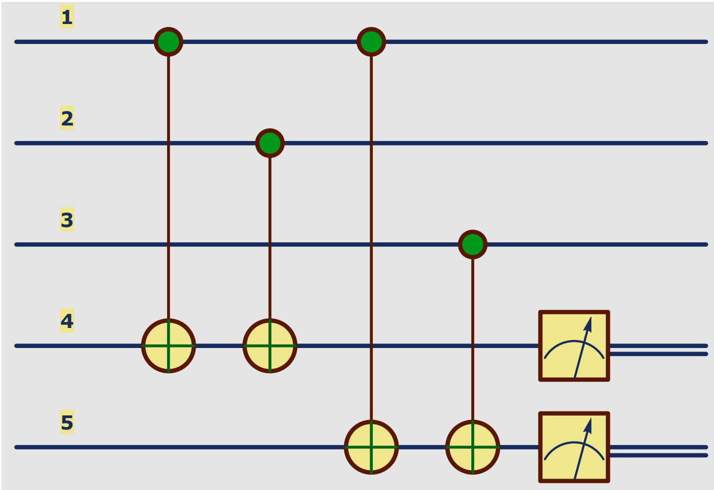
Therefore, an arbitrary single qubit state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ is mapped to,

$$|\varphi\rangle \to \alpha|000\rangle + \beta|111\rangle = |\varphi\rangle_L \tag{2}$$

This mapping is achieved by the following quantum circuit, where input 1 is $|\varphi\rangle$, and the other two are ancilla qubits.



The three qubit code is protected from single qubit flip error. We need first to “measure” where the error occurs and then correct it. However, we cannot directly measure the qubits. $|\varphi\rangle_L$ is a superposition of $|000\rangle$ and $|111\rangle$ in general. Any measurement will destroy the superposition. We use additional two ancilla qubits to extract syndrome information from the data block without discriminating the exact state of any qubit. This is achieved by the following quantum gate.



The syndrome are listed below.

```
Block[{label, data},
 data = {{"Error location", "|final state⟩⊗|ancilla⟩"},
   {"No error", "α|000⟩|00⟩+β|111⟩|00⟩"}, {"Qubit 1", "α|100⟩|11⟩+β|011⟩|11⟩"},
   {"Qubit 2", " α|010⟩|10⟩+β|101⟩|10⟩"}, {"Qubit 3", "α|001⟩|01⟩+β|110⟩|01⟩"}};
 Grid[data, Frame → All]]
```

Error location	|final state⟩⊗|ancilla⟩
No error	α|000⟩|00⟩+β|111⟩|00⟩
Qubit 1	α|100⟩|11⟩+β|011⟩|11⟩
Qubit 2	α|010⟩|10⟩+β|101⟩|10⟩
Qubit 3	α|001⟩|01⟩+β|110⟩|01⟩

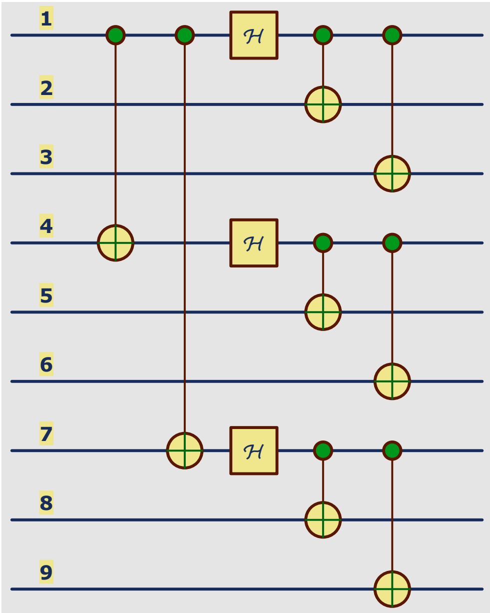
Three qubit code has its limitation. It cannot detect qubit flip error occurs more than one place. For example, qubit 1&2 flip error will be detected as qubit 3 flip error. And it cannot detect phase shift error.

▪ Nine qubit code

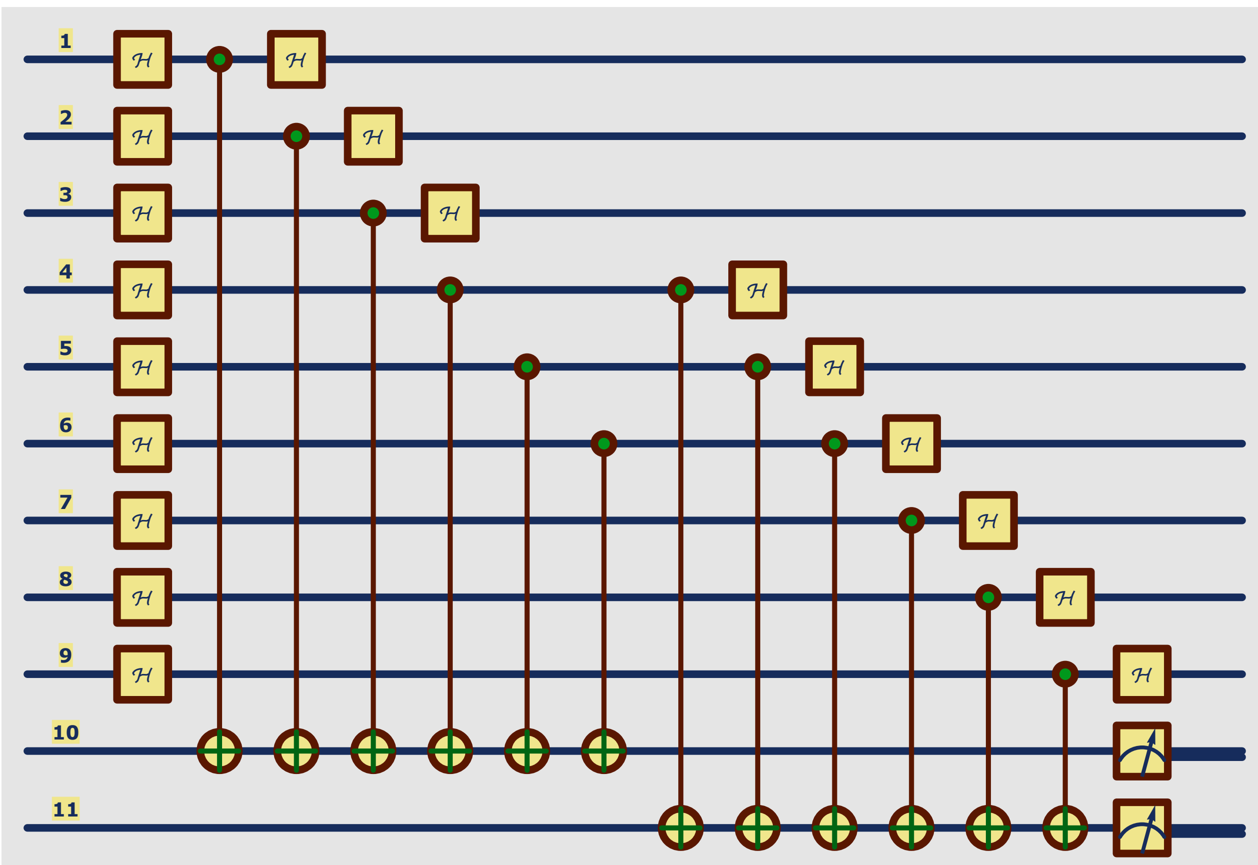
In nine qubit code, the basis for logic code are

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \quad (3)$$

The preparation can be achieved using



The X-error correction is the same as three-qubit code. For Z-error correction, we use the following quantum circuit,



The last two qubits are the ancilla qubits to measure the syndrome.

Properties of any quantum code

Here, we discuss the sufficient and necessary condition for a quantum code to detect and correct error $\{E_i\}$, which is not necessary to form a group. In order for the code to correct two errors E_1 and E_2, we must always be able to distinguish error E_1 acting on basis code-word $|\varphi_j\rangle$ from error E_2 acting on a different basis code word $|\varphi_j\rangle$. Therefore,

$$\langle\varphi_i|\, E_1^{\dagger} E_2\, |\varphi_j\rangle = 0 \tag{4}$$

We can always include identity to the errors, because we don't want to confuse with one error to a basis code word with another basis code word.

However, only Eq.(4) is insufficient to guarantee a quantum error-correcting code. When we make a measurement to find out the error, we must not learn anything about the actual state in the code space, or it will collapse the code word.

We learn the error by measuring the quantity about $\langle\varphi_i|\, E_a^{\dagger} E_b\, |\varphi_i\rangle$, therefore this quantity must be the same for all the state basis, i.e. $\langle\varphi_i|\, E_a^{\dagger} E_b\, |\varphi_i\rangle = \langle\varphi_j|\, E_a^{\dagger} E_b\, |\varphi_j\rangle$. Combine those two conditions, we arrived at the sufficient and necessary condition for error correction:

$$\langle\varphi_i|\, E_a^{\dagger} E_b\, |\varphi_j\rangle = \delta_{ij}\, C_{ab}, \tag{5}$$

where i and j run over all the basis in codeword space; a and b run over all possible errors. This condition was first derived by Knill, Laflamme[2], and Bennett[3].

Stabilizer formalism

Stabilizer formalism is an efficient way to describe quantum error correction code that is independent of state representation. Its main idea is to describing quantum states in terms of operators rather than the states.

Let's revisit the nine-qubit code. To detect a bit flip error, we compared the first and second qubits, and then compared the first and third qubits. This is "equivalent" to measure the eigenvalues of $\sigma_{z1}\,\sigma_{z2}$ and $\sigma_{z1}\,\sigma_{z3}$. Similarly, to detect a sign error, we compared the signs of first and second blocks of three and the first and third block. This is "equivalent" to measure the eigenvalues of $\sigma_{x1}\,\sigma_{x2}\,\sigma_{x3}\,\sigma_{x4}\,\sigma_{x5}\,\sigma_{x6}$ and $\sigma_{x1}\,\sigma_{x2}\,\sigma_{x3}\,\sigma_{x7}\,\sigma_{x8}\,\sigma_{x9}$.

Generator	Pauli String
M_1	*Z Z I I I I I I I*
M_2	*Z I Z I I I I I I*
M_3	*I I I Z Z I I I*
M_4	*I I I Z I Z I I I*
M_5	*I I I I I I Z Z I*
M_6	*I I I I I I Z I Z*
M_7	*X X X X X X I I I*
M_8	*X X X I I I X X X*

The above table is the generators of the group G, that $|0\rangle_L$ and $|1\rangle_L$ are eigenvector of all these generators with eigenvalues +1. This group G is called the stabilizer of the code.

We see if there is a bit flip error that happened on bit 2, whose error can be generated by $E_2 = \sigma_{x2}$, then $\{E_2, M_2\} = 0$, and E_2will commute with all other generators. Therefore,

$$M\,E\,|\varphi\rangle = -E\,|\varphi\rangle, \tag{6}$$

So E$|\varphi\rangle$ is an eigenvector of M with eigenvalue -1, and to detect E, we only need to measure M.The distance of nine qubit code is three. For instance, we can find operator $\sigma_{x1}\,\sigma_{x2}\,\sigma_{x3}$ that

$$\langle 0|_L \sigma_{x1}\sigma_{x2}\sigma_{x3} |0\rangle_L = 1 \quad \langle 1|_L \sigma_{x1}\sigma_{x2}\sigma_{x3} |1\rangle_L = -1 \tag{7}$$

Need more understanding on distance between quantum code.

Now, we formulate stabilizer code from group theory point of view.The Pauli group, $\mathcal{P}$, contains

$$\mathcal{P} = \{\pm\sigma_I, \pm i\sigma_I; \pm\sigma_X, \pm i\sigma_X; \pm\sigma_Y, \pm i\sigma_Y; \pm\sigma_Z, \pm i\sigma_{IZ}\} \tag{8}$$

For n-qubits, we have $\mathcal{P}_N = \mathcal{P}^{\otimes N}$. There are some general properties of $\mathcal{P}_N$:

- **Each of element is either Hermitian, $M^\dagger = M$, or anti-Hermitian, $M^\dagger = -M$.**
- **Each of element is either unitary: $M^\dagger M = I$, or anti-unitary: $M^\dagger M = -I$.**

In general, the stabilizer $\mathcal{S}$ is some Abelian subgroup of $\mathcal{P}_N$ and the coding space $\mathcal{T}$ is the space of vectors that are fixed by $\mathcal{S}$. If there are (N-k) generators of $\mathcal{S}$, then the coding space $\mathcal{T}$ has dimension 2^k, which means there are k qubits can be encoded. And a quantum code with stabilizer $\mathcal{S}$ will detect all the errors $\mathcal{E}$ that are either in $\mathcal{S}$ or anti-commute with some element of $\mathcal{S}$.

The error syndrome for a stabilizer code is defined using mapping $f_M : \mathcal{P}_N \to \mathbb{Z}_2$,

$$f_M(E) = 0, \text{ if } [M, E] = 0; \ 1 \text{ if } \{M, E\} = 0 \tag{9}$$

and $f(E) = (f_{M_1}(E), f_{M_2}(E), \ldots, f_{M_{n-k}}(E))$ is a (n-k)-bit binary number. $f(E_a) = f(E_b)$, iff $f(E_a E_b) = 0$. So for non-degenerate code, f(M) is different for each correctable error E.

The elements of normalizer $\mathcal{N}(\mathcal{S})$ minus $\mathcal{S}$, i.e. $\mathcal{N}(\mathcal{S})$-$\mathcal{S}$, move the codewords around within $\mathcal{T}$, so they have a natural interpretation as encoded operations on the codewords.

Some examples

Five-qubit code

Element	Operator
M_1	$\sigma_x\ \sigma_z\ \sigma_z\ \sigma_x\ I$
M_2	$I\ \sigma_X\ \sigma_z\ \sigma_z\ \sigma_x$
M_3	$\sigma_x\ I\ \sigma_x\ \sigma_z\ \sigma_z\ \sigma_x$
M_4	$\sigma_z\ \sigma_x\ I\ \sigma_x\ \sigma_z$
$\overline{X}$	$\sigma_x\ \sigma_x\ \sigma_x\ \sigma_x\ \sigma_x$
$\overline{Z}$	$\sigma_z\ \sigma_z\ \sigma_z\ \sigma_z\ \sigma_z$

■ Eight-qubit code

Element	Operator
M_1	*X X X X X X X X*
M_2	*Z Z Z Z Z Z Z Z*
M_3	*I X I X Y Z Y Z*
M_4	*I X Z Y I X Z Y*
M_5	*I Y X Z X Z I Y*
$\overline{X_1}$	*X X I I I Z I Z*
$\overline{X_2}$	*X I X Z I I Z I*
$\overline{X_3}$	*X I I Z X Z I I*
$\overline{Z_1}$	*I Z I Z I Z I Z*
$\overline{Z_2}$	*I I Z Z I I Z Z*
$\overline{Z_3}$	*I I I I Z Z Z Z*

Reference

[1] arXiv: 0905.2794v4(2013)

[2] "A theory of quantum error-correcting codes", Phys. Rev. A 55, 900(1997)

[3] "Mixed state entanglement and quantum error correction", Phys. Rev. A 54, 3824(1996)